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U-duality and BPS spectrum of super Yang-Mills theory and M-theory

Chris M. Hull

Physics Department, Queen Mary and Westfield College,
Mile End Road, London E1 4NS, U.K.
Laboratoire de Physique Théorique, Ecole Normale Supérieure,
24 Rue Lhomond, 75231 Paris Cedex 05, France.
E-mail: C.M.Hull@qmw.ac.uk

Abstract: It is shown that the BPS spectrum of Super-Yang-Mills theory on $T^d \times \mathbb{R}$, which fits into representations of the U-duality group for M-theory compactified on T^d, in accordance with the matrix-theory conjecture, in fact fits into representations of the U-duality group for M-theory on T^{d+1}, with the extra dualities realised as generalised Nahm transformations. The spectrum of BPS M-branes is analysed, new branes are discussed and matrix theory applications described.

Keywords: M(atrix) Theories, M-Theory, String Duality, Supersymmetry and Duality.

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Contents

1. Introduction

A remarkable relation has been proposed between Super-Yang-Mills (SYM) theory and M-theory [1]-[17]. M-theory in the infinite momentum frame on $R^{10} \times S^1$, in the limit in which the radius of the 'longitudinal' circle becomes infinite, $R_{11} \to \infty$, corresponds to one dimensional $U(N)$ SYM in the limit $N \to \infty$. M theory compactified on a d torus, i.e. on $T^d \times R^{10-d} \times S^1$, in the limit $R_{11} \to \infty$, corresponds to a theory whose low-energy limit is the $N \to \infty$ limit of $d+1$ dimensional $U(N)$ SYM on $\tilde{T}^d \times \mathbb{R}$ where $\tilde{T}^d$ is the dual torus and $\mathbb{R}$ is the time direction [1]-[17]. The $d+1$ dimensional SYM is given by dimensional reduction of 10-dimensional SYM. For $d \leq 3$, the SYM theory is a consistent quantum theory and gives the full matrix theory description of the M-theory. For $d > 3$, however, extra degrees of freedom become important at short distances and strong coupling, so that SYM is only an effective theory. For $d = 4$, the full theory is the $5+1$ dimensional (2,0) tensor multiplet theory [9]-[12], and for $d = 5$ it is a $5+1$ dimensional non-critical string theory [8]-[12]. In particular, M-theory compactified on a d torus has U-duality group $E_d(\mathbb{Z})$ [20] (where $E_5 = SO(5,5)$, $E_4 = SL(5)$, $E_3 = SL(3) \times SL(2)$, $E_2 = SL(2) \times \mathbb{R}$ and $E_d = E_{d,d}$ for $d = 6,7,8$), and this is a manifest symmetry of the matrix theory, at least for $d \leq 5$. The $SL(d,\mathbb{Z})$ subgroup of $E_d(\mathbb{Z})$ is the geometrical symmetry associated with the torus $\tilde{T}^d$. For $d = 3$, the duality group is $SL(3) \times SL(2)$, and the extra $SL(2,\mathbb{Z})$ arises from the S-duality of 3+1 dimensional SYM.

It has been proposed that M-theory on $T^d \times R^{10-d} \times S^1$ for *finite* R_{11}, in the discrete light cone gauge, is related to $U(N)$ SYM on $\tilde{T}^d \times \mathbb{R}$ for *finite* N [3]. The case in which the extra circle is null can be thought of as an infinite boost limit of the space-like reduction [16, 17], and the fact that M-theory compactified on $T^d \times S^1$ where the extra circle is space-like has a U-duality group E_{d+1} suggests that the $d+1$ dimensional SYM on $\tilde{T}^d \times \mathbb{R}$ should have $E_{d+1}(\mathbb{Z})$ (or perhaps a discrete subgroup of a contraction of E_{d+1}), and should have more than just the the $E_d(\mathbb{Z})$ symmetry of the large N limit.

In [19], the duality groups for null reductions of 11-dimensional supergravity will be considered. In [13], evidence was given that for $d = 3$ there is E_4 symmetry instead of the expected E_3. For finite N, it was argued that the expected $E_3 = SL(3) \times SL(2)$ symmetry is enhanced to $SL(5)$, with the extra duality symmetries realised as Nahm-type transformations that interchange the rank N of the gauge group with electric or magnetic fluxes. It was shown that the E_3 representations of BPS states fit into representations of E_4.

In this paper, evidence will be given that the BPS spectrum of SYM indeed fits into representations of E_{d+1} for other values of d (if a matrix theory exists for such d), based on and generalising the results of [13, 14]. Similar results have also been obtained by Blau and O'Loughlin [18]. The information about BPS states so obtained will then be used to re-examine the brane spectrum of M-theory. Indeed, much of the analysis here also applies to M-theory compactification on a conventional (space-like) torus. Starting from the known 0-brane spectrum (from wrapped M2 and M5 branes, and from pp-waves etc) and acting with U-duality, the expected multiplets are obtained for compactification to 5 dimensions or higher, but for lower dimensions new exotic branes are generated. In four dimensions, the extra branes are just the wrapped Kaluza-Klein (KK) monopole 6-branes, but for 3 dimensions, many extra branes are needed to complete the 0-brane multiplet, which is found to be a **248** of E_8. Some of the extra branes were anticipated in [22] and references therein, while others were first proposed in [14].

For $d \geq 6$, it is not known whether a matrix theory limit exists, or whether there is a consistent quantum theory with a $d + 1$ dimensional SYM effective low-energy description, and the results here will only apply for those dimensions in which these theories do exist. For $d = 3$, SYM on $T^3 \times \mathbb{R}$ has $SL(3;\mathbb{Z})$ torus symmetry, together with a conjectured S-duality and conjectured duality under Nahm transformations, generating an $SL(5;\mathbb{Z})$ symmetry [13]. The theory on $T^d \times \mathbb{R}$, if it exists, should reduce to SYM on $T^3 \times \mathbb{R}$ at low energies and in any limit in which a T^{d-3} shrinks, leaving an effective theory with approximate S-duality and Nahm-duality symmetries. In [14], it was shown that if the S-duality of this limit extends to an exact symmetry of the full $d+1$ dimensional theory, then the S-dualities together with the $SL(d;\mathbb{Z})$ torus symmetries together generate the U-duality group $E_d(\mathbb{Z})$. Here, this will be extended to show that if the Nahm dualities also extend to $d+1$ dimensional exact symmetries, then $E_{d+1}(\mathbb{Z})$ is generated.

In [3, 16, 17], the $d+1$ $U(N)$ SYM theory arises from the discrete light cone gauge quantization of M-theory on the product of T^d with a null circle, for fixed quantized longitudinal momentum, which is proportional to N. Including all possible momenta, which is required for Lorentz invariance, requires the consideration of a larger theory, with a quantum number N, such that the sector with a given N has a $U(N)$ SYM description. Moreover, an $N = 0$ sector should presumably be included, even though it would not have a conventional SYM description.

The Nahm dualities act in SYM by interchanging the rank N with electric and magnetic fluxes, so that it relates BPS states of a $U(N)$ theory with flux N' to states of a dual $U(N')$ theory with flux N. It is not a symmetry of a given SYM theory, but could be a symmetry of the larger 'matrix' theory which includes sectors for all values of N. It will also include transformations relating $N \neq 0$ sectors with the $N = 0$ sector.

The Nahm dualities act in M-theory by interchanging the longitudinal radius R_{11} with the radius R_i of a space-like circle, and so are relevant to the Lorentz invariance of the theory. The U-dualities also act on the supergravity background fields, which correspond to torus moduli or to generalised coupling constants of the SYM theory [24]. In particular, the Nahm transformations in $E_{d+1}(\mathbb{Z})/E_d(\mathbb{Z})$ generate non-trivial anti-symmetric tensor gauge-fields on the null torus and for such backgrounds, non-commutative geometry plays a role [25]. These situations are best described by SYM on a non-commutative torus [25], and a deeper understanding of such situations is clearly needed. In the following, we will ignore such considerations and preceed formally.

2. U-duality and the BPS Spectrum of SYM

We now review some of the results of [14], in which the BPS spectrum of SYM on $\tilde{T}^d \times \mathbb{R}$ was shown to fit into representations of E_d (with some subtleties for $d \geq 7$). We follow the notation of [14]; $\tilde{T}^d$ is taken to be a rectangular torus with circles of radii s_i, $i = 1, \ldots, d$, and the SYM coupling is g. This corresponds to M-theory on a rectangular torus $T^d \times S^1$, with circles of radii R_i, R_{11}, and coupling given by the $D = 11$ Planck length l_p. The volumes of the d-tori $\tilde{T}^d, T^d$ are

$$V_R = \prod_{i=1}^{d} R_i, \qquad V_s = \prod_{i=1}^{d} s_i\,. \tag{2.1}$$

The relationship between the two sets of variables is

$$\begin{aligned} s_i &= \frac{l_p^3}{R_{11} R_i} \\ g^2 &= \frac{l_p^{3(d-2)}}{R_{11} V_R}\,. \end{aligned} \tag{2.2}$$

The SYM theory on $\tilde{T}^d \times \mathbb{R}$ is invariant under the manifest $SL(d, \mathbb{Z})$, and this is reduced to the permutation group on the d circles by restricting the torus to be rectangular. If three of the circles are much larger than the others, the theory will behave like $3 + 1$ dimensional SYM at low energies and should have an approximate Montonen-Olive $SL(2, \mathbb{Z})$ symmetry. In [14], it was supposed that this should extend to a symmetry for all values of the radii, at least at low energies, so that for any 3 of the d circles, there is a Montonen-Olive transformation S_{ijk}. The transformation S_{ijk}

Yang-Mills Energy	M-Theory Mass	Degeneracy
$\frac{1}{s_i}$	$\frac{R_{11} R_i}{l_p^3}$	d
$\frac{s_{i_1} \cdots s_{i_{d-4}}}{g^2}$	$\frac{R_{11} R_{j_1} \cdots R_{j_4}}{l_p^6}$	$\frac{d!}{4!(d-4)!}$
$\frac{V_s}{g^4} \frac{s_{i_1} \cdots s_{i_{d-6}}}{s_i}$	$\frac{R_{11} R_i^2 R_{j_1} \cdots R_{j_5}}{l_p^9}$	$6 \frac{d!}{6!(d-6)!}$
$\frac{V_s^2}{g^6} \frac{s_{i_1} \cdots s_{i_{d-7}}}{s_i s_j s_k}$	$\frac{R_{11} R_i^2 R_j^2 R_k^2 R_{j_1} \cdots R_{j_4}}{l_p^{12}}$	$35 \frac{d!}{7!(d-7)!}$

Table 1: Momentum multiplet states.

Yang-Mills Energy	M-Theory Mass	Degeneracy
$\frac{g^2 s_i^2}{N V_s}$	$\frac{1}{R_i}$	d
$\frac{V_s}{N g^2 s_i^2 s_j^2}$	$\frac{R_i R_j}{l_p^3}$	$\frac{d!}{2!(d-2)!}$
$\frac{V_s s_{i_1}^2 \cdots s_{i_{d-5}}^2}{N g^6}$	$\frac{R_{j_1} \cdots R_{j_5}}{l_p^6}$	$\frac{d!}{5!(d-5)!}$
$\frac{V_s^3}{N g^{10}} \frac{s_{i_1}^2 \cdots s_{i_{d-7}}^2}{s_i^2}$	$\frac{R_i^2 R_{j_1} \cdots R_{j_6}}{l_p^9}$	$7 \frac{d!}{7!(d-7)!}$
$\frac{V_s^5}{N g^{14}} \frac{s_{i_1}^2 \cdots s_{i_{d-8}}^2}{s_i^2 s_j^2 s_k^2}$	$\frac{R_i^2 R_j^2 R_k^2 R_{j_1} \cdots R_{j_5}}{l_p^{12}}$	$56 \frac{d!}{8!(d-8)!}$

Table 2: Flux multiplet states.

acts on the M-theory variables as [14]

$$R_i \to \frac{l_p^3}{R_j R_k}\,, \qquad R_j \to \frac{l_p^3}{R_k R_i}\,, \qquad R_k \to \frac{l_p^3}{R_k R_i}\,, \tag{2.3}$$

with all other radii unchanged. This is a T-duality combined with a permutation (from the Weyl group of $SL(d, \mathbb{Z})$). The action of these transformations on the SYM variables is given in [14]. The permutation group together with the Montonen-Olive transformations S_{ijk} generate the Weyl group of $E_d(\mathbb{Z})$ and these were used to build up U duality multiplets $E_d(\mathbb{Z})$. The momentum multiplet was generated by acting with U-duality on the Kaluza-Klein mode with momentum in the i'th direction and energy

$$E = \frac{1}{s_i}\,, \tag{2.4}$$

while the flux multiplet was obtained by acting with states carrying electric flux in the i'th direction and energy

$$E = \frac{g^2 s_i^2}{N V_s}\,. \tag{2.5}$$

This gave rise to an infinite set of states, of which only a finite number are relevant for $d \leq 8$. The first few states in the momentum and flux multiplets have [14] masses and multiplicities shown in Tables 1, 2.

d	Mom Multiplet: $SL(d,\mathbb{R})$ Reps	Mom Multiplet: E_d Reps	Flux Multiplet: $SL(d,\mathbb{R})$ Reps	Flux Multiplet: E_d Reps
3	3	(3,1)	3+3	$(3',2)$
4	4+1	5	4+6	10
5	5+5	10	10+5+1	16
6	6+15+6	27	6+15+6	$27'$
7	7+35+42+35+7	$126 \subset 133$	7+21+21+7	56
8	8+70+168+...	$2160 \subset 3875$	8+28+56+56+56+28+8	$240 \subset 248$

Table 3: $SL(d,\mathbb{R})$ and E_d Representations of Momentum and Flux multiplets.

The states in the second row of the flux multiplet table carry magnetic flux in the i, j directions and arise in M-theory from wrapped membranes, while the states in the second row of the momentum multiplet table are $d-4$ branes arising from SYM instantons [14], or from longitudinal M5-branes in the M-theory picture. The SYM interpretation of the other states is not well understood. The momentum and flux multiplets of states fit into representations of $SL(d,\mathbb{R})$ and E_d [14] shown in Table 3.

For $d = 7, 8$, the states obtained in this way do not appear to fit into complete representations of E_7, E_8, suggesting that there should be extra states missing from this list if U-diality is to be a symmetry of the BPS spectrum. For example, for the $d = 7$ momentum mulitiplet, the decomposition of the **133** of E_7 into $SL(7)$ representations is $7 + 35 + 48 + 1 + 35 + 7$ instead of the $7 + 35 + 42 + 35 + 7$ in table 3. Thus the **42** should be part of a **48+1** representation of $SL(7)$. The **42** corresponds to states with mass

$$\frac{V_R R_{11}}{l_p^9}\frac{R_i}{R_j}, \tag{2.6}$$

for $i \neq j$. The extra 7 states must arise from states with the same mass formula, but with $i = j$ for each of the seven values of $i = j$, as they must be related to those for $i \neq j$ by $SL(7)$ transformations. This gives seven states, each with mass

$$\frac{V_R R_{11}}{l_p^9} = \frac{R_{i_1} \ldots . R_{i_7} R_{11}}{l_p^9}. \tag{2.7}$$

Similarly, the $d = 8$ fluxes multiplet has **56** states with masses given by

$$\frac{V_R}{l_p^9}\frac{R_i}{R_j} \tag{2.8}$$

for $i \neq j$ which must in fact fit into a **63+1** of $SL(8)$. Including states of this form with $i = j$ gives a further 8 states, each with mass

$$\frac{V_R}{l_p^9} = \frac{R_{i_1} \ldots . R_{i_8}}{l_p^9}. \tag{2.9}$$

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Such 'M8-brane' states were first suggested in [14]. Similarly, the $d = 8$ momentum multiplet is completed to a **3875** of E_8. That such extra states complete the E_7, E_8 representations was also found by the authors of [14].[1]

This will now be generalised to include the Nahm transformations of [13]. Again, if three of the circles are much larger than the others, the theory will behave like $3+1$ dimensional SYM at low energies and should have in addition the Nahm symmetry of [13]. Assuming that for low energies this extends to a symmetry for all values of the s_i, as was done above for the Montonen-Olive symmetry, we obtain a further discrete transformation for any three circles that, when combined with the transformations above in (the Weyl group of) E_d, generates (the Weyl group of) E_{d+1}. Then repeating the analysis of [14] but with the extra Nahm transformations, we should find SYM states fitting into multiplets of E_{d+1}. While the Nahm transformations are complicated to apply on the SYM variables and will not be given explicitly here, they act simply on the M-theory variables: they correspond to the permutation swapping R_{11} with R_i for some i [13]. Thus they enlarge the the Weyl group of $SL(d)$ to that of $SL(d+1)$. Acting on the flux multiplet given in table 2 then gives extra states, which can be organised according to the power of R_{11} in the M-theory mass. There is one state with mass

$$\frac{1}{R_{11}}, \tag{2.10}$$

which corresponds to the D0-brane in the type IIA string theory limit. The first few states with $M \propto R_{11}^2$ have the following M-theory masses:

$$\begin{aligned} &\frac{R_{11}^2 R_{j_1} \cdots R_{j_6}}{l_p^9}, \qquad \frac{R_{11}^2 R_j^2 R_k^2 R_{j_1} \cdots R_{j_5}}{l_p^{12}}, \\ &\frac{R_{11}^2 R_j R_k R_{j_1}^2 \cdots R_{j_5}^2}{l_p^{15}}, \qquad \frac{R_{11}^2 R_i^3 R_{j_1}^2 \cdots R_{j_6}^2}{l_p^{18}}, \ldots \end{aligned} \tag{2.11}$$

The first of these gives the D6-brane in the type IIA string theory limit. The first few states with $M \propto R_{11}^3$ have masses:

$$\frac{R_{11}^3 R_{j_1} \cdots R_{j_8}}{l_p^{12}}, \qquad \frac{R_{11}^3 R_{k_1} \cdots R_{k_5} R_{j_1}^2 \cdots R_{j_3}^2}{l_p^{15}}, \qquad \frac{R_{11}^3 R_{j_1}^2 \cdots R_{j_7}^2}{l_p^{18}}, \ldots \tag{2.12}$$

The first of these is the D8-brane. There is a state with mass linear in R_{11}, which corresponds to a longitudinal version of the 8-brane (2.9), with mass

$$\frac{R_{11} R_{j_1} \cdots R_{j_7}}{l_p^9}. \tag{2.13}$$

The flux multiplet is the set of all states with mass independent of R_{11}, generated by acting on a momentum state with mass $1/R_i$ with E_d and the momentum multiplet is

[1] This is contained in a Note Added to the version of [14] to be published in Nucl. Phys. B. I would like to thank David Kutasov for informing me of this.

d	E_d: Momentum	E_d: Flux	E_d: Extra	E_{d+1}: All
3	(3,1)	$(3',2)$	(1,1)	10
4	5	10	1	16
5	10	16	1	27
6	27	$27'$	1+1	56
7	133	56	56+1+1+1	248
8	3875	248	1+248+...	

Table 4: Representations of E_{d+1}.

the set of all states with mass linear in R_{11}, generated by acting on a longitudinal membrane state with mass $R_i R_{11}/l_p^3$ with E_d. Similarly, there is an E_d multiplet generated by acting with E_d on the D6-brane with mass

$$\frac{R_{11}^2 R_{j_1} \cdots R_{j_6}}{l_p^9} . \tag{2.14}$$

The first few members of this multiplet are given in (2.10)-(2.13), and form a singlet of E_6 for $d = 6$, a **56** of E_7 for $d = 7$ and a **248** of E_8 for $d = 8$. Similarly, there is a multiplet with mass proportional to R_{11}^3, the first few members of which are listed in (2.12), giving a singlet in $d = 7$ etc. There are further multiplets with mass proportional to R_{11}^n for $n > 3$ which first contribute for $d = 8$. The state (2.10) is an E_d singlet which appears for all d.

These states can be combined with the states from table 3 to have the right counting to fit into representations of E_{d+1} shown in Table 4.

They must certainly fit into representations of the $SL(d+1)$ subgroup of E_{d+1} which acts naturally on $T^d \times S^1$ irrespective of whether the circle is null or space-like, and they also transform as representations of E_d. For $d = 3$, they fit into representations of $SL(5; \mathbb{Z})$ (with the provisos given earlier). Then it follows they must in fact transform as the representations of E_{d+1} in table 4.

Thus the BPS states of the $d + 1$ dimensional SYM theory on $T^d \times \mathbb{R}$ fit into representations of $E_{d+1}(\mathbb{Z})$, and these should extrapolate to BPS states of the full matrix theory. These correspond to BPS states of M-theory compactified on T^{d+1} (with one null circle), which includes some novel states for $d \geq 7$, some of which were discussed in [14]. For M-theory compactified on T^d, the flux multiplet states of table 2 give M-theory states with masses

$$\frac{1}{R_i}, \quad \frac{R_i R_j}{l_p^3}, \quad \frac{R_{j_1} \cdots R_{j_5}}{l_p^6}, \quad \frac{R_i^2 R_{j_1} \cdots R_{j_6}}{l_p^9},$$
$$\frac{R_{j_1} \cdots R_{j_8}}{l_p^9}, \quad \frac{R_i^2 R_j^2 R_k^2 R_{j_1} \cdots R_{j_5}}{l_p^{12}}, \quad \frac{R_i^3 R_{j_1} \cdots R_{j_8}}{l_p^{12}},$$
$$\frac{R_i R_j R_{j_1}^2 \cdots R_{j_6}^2}{l_p^{15}}, \quad \frac{R_i^3 R_{k_1} \cdots R_{k_5} R_{j_1}^2 \cdots R_{j_3}^2}{l_p^{15}}, \quad \frac{R_i^3 R_{j_1}^2 \cdots R_{j_7}^2}{l_p^{18}}, \ldots \tag{2.15}$$

where $i = 1, ..., d$. The list is infinite, but only a finite number of these states occur for $d \leq 8$, and (2.15) contains all states occuring for $d \leq 8$. We have seen that these flux multiplet states combine with the momentum multiplet states and the extra states (2.10)-(2.13) to give states fitting into representations of E_{d+1} that are given by precisely the *same* list as in (2.15), but with the indices running over $d+1$ values $i = 1, ..., d, 11$ to include the longitudinal direction R_{11}. The same list of masses would be obtained for the 'flux multiplet' of states in the compactification of M-theory on a *space-like* torus T^{d+1}. Indeed, the same list could have been generated by acting with $E_{d+1}(\mathbb{Z})$ on the states with $M = 1/R_i$. Thus U-duality gives the complete multiplet of M-theory states which includes the expected wrapped M-branes etc, together with some unexpected new states for $d \geq 7$.

In the following, it will be useful to refer to a state with mass dependence proportional to

$$(R_{i_1} \cdots R_{i_p})^{\alpha} (R_{j_1} \cdots R_{j_q})^{\beta} ... (R_{k_1} \cdots R_{k_p})^{\gamma} , \tag{2.16}$$

with $\alpha < \beta < .... < \gamma$ as a $(p^{\alpha}, q^{\beta}, ..., r^{\gamma})$-brane and, if $\alpha = 1$, to refer to a $(p^1, q^{\beta}, ..., r^{\gamma})$-brane as a $(p, q^{\beta}, ..., r^{\gamma})$-brane (i.e. to drop the exponent $\alpha = 1$), while a (p^1)-brane with $p \geq 0$ will be referred to as a p-brane, in the usual way. A state in M-theory with mass dependence (2.16) will have a mass proportional to l_p^{-a} where $a = \alpha p + \beta q + ... + \gamma r + 1$. Such states will appear in a compactification on T^d only for $d \geq p + q + ... + r$. The states in the list (2.15) are then $(p^1, q^{\beta}, ..., r^{\gamma})$-branes with $(p^1, q^{\beta}, ..., r^{\gamma})$ given respectively by

$$\begin{array}{llll} (-1)\,, & 2\,, & 5\,, & (6, 1^2)\,, \\ 8, & (5, 3^2)\,, & (8, 1^3) & \\ (2, 6^2)\,, & (5, 3^2, 1^3)\,, & (7^2, 1^3), ... & \end{array} \tag{2.17}$$

3. The BPS Brane Spectrum of M-Theory and Type II String Theory

We now turn to the interpretation of the BPS states (2.15), (2.17) in M-theory and in type II string theory. We will start by considering M-theory compactified on a space-like torus T^d, which has 0-brane states given by (2.15) with $i = 1, ..., d$. In the limit in which one of the radii becomes small, R_s say, then a weakly coupled IIA string theory emerges with string coupling g_s given by

$$g_s = \frac{R_s}{l_s} , \tag{3.1}$$

where the string length is

$$l_s^2 = \frac{l_p^3}{R_s} . \tag{3.2}$$

The weakly coupled string limit is then obtained by taking $g_s \to 0$ while keeping g_s fixed. Different string theory limits emerge for different choices of R_k.

The states with $M = 1/R_m$ are Kaluza-Klein modes (or pp-waves) carrying momentum in the m'th direction. Such states with $m = 1, ..., d$ generated the flux multiplet in table 2, while the state with $M = 1/R_{11}$ was the first 'extra' state added in (2.10) to obtain E_{d+1} representations. It is a state with longitudinal momentum, corresponding to a D0-brane of type IIA. 2-brane states with $M = R_m R_n / l_p^3$ are membranes wrapped around the m, n directions. States with $M = R_{m_1} ... R_{m_5} / l_p^6$ are wrapped 5-branes. As is well known, in the type IIA limit, the 2-brane gives a string or D2-brane, the 5-brane gives a 5-brane or D4-brane and the pp-waves give pp-waves or D0-branes.

The $(6, 1^2)$ states with

$$M = \frac{R_1 \dots R_6 R_7^2}{l_p^9}, \tag{3.3}$$

were considered in [14]; we give a related interpretation here. These are Kaluza-Klein monopoles, or G-branes of M-theory [21, 22]. A KK monopole or G6-brane in M-theory is the space-time $\mathbb{R}^{6,1} \times TN$ where TN denotes Taub-NUT space. This can be wrapped to give the solution $\mathbb{R} \times T^6 \times TN$ and depends on the 6 radii $R_1, \dots, R_6$ of T^6 and the asymptotic radius R_7 of the S^1 fibre of TN, which is proportional to the NUT parameter N. These states carry the charge discussed in [22] which occurs in the supersymmetry algebra, and the mass is given by this charge. It is proportional to $NR_1 \dots R_6 R_7$ [22], which gives (3.3) as $N \propto R_7$.

For the $(6, 1^2)$ states with mass (3.3), taking R_7 as the radius R_s in (3.2) corresponding to the string coupling g_s gives a state with mass

$$M = \frac{R_1 \dots R_6}{g_s l_s^7} \tag{3.4}$$

and is the D6-brane of the IIA theory [14]. If R_1 corresponds to g_s, the state is a $(5, 1^2)$-brane with

$$M = \frac{R_2, \dots R_6 R_7^2}{g_s^2 l_s^8} \tag{3.5}$$

and is the KK monopole or G5-brane of the IIA theory, related by T-duality to the NS 5-brane of the IIB theory [20, 22]. For g_s corresponding to another direction, R_8 say, the mass is

$$M = \frac{R_1 \dots R_6 R_7^2}{g_s^3 l_s^9}, \tag{3.6}$$

so that this is a $(6, 1^2)$-brane of type IIA theory. T-duality in the R_8 direction gives a decompactification of the dual coordinate, $R_8' \to \infty$ and gives a IIB-brane with

$$M = \frac{R_1 \dots R_6 R_7}{g_s^3 l_s^8}. \tag{3.7}$$

This is the (1,0) 7-brane of the type IIB theory which is related by $SL(2, \mathbb{Z})$ to the (0,1) D7-brane. The fact that this has $1/g^3$ dependence was shown in [23], where the

g dependence of branes in string theory was calculated. We now review and elaborate on this.

In the IIB theory, consider a p-brane whose energy density is proportional to g^{-a} at weak coupling; for a D-brane with $p = 1, 3, 5, 7, 9$, $a = 1$, while for a NS 5-brane or G5-brane (KK monopole), $a = 2$. The IIB theory has 1,5,7 and 9 branes labelled by 2 charges (p, q) which transform under $SL(2, \mathbb{Z})$ [22]. The (0,1) branes are D-branes, while the (1,0) 1-brane and 5-brane couple to the 2-form in the Neveu-Schwarz sector. Then, under the $SL(2, \mathbb{Z})$ duality that takes $g \to \tilde{g} \equiv 1/g$, these transform to states with energy density proportional to $\tilde{g}^{-\tilde{a}}$ where $\tilde{a} = (p + 1)/2 - a$. For a D-brane, $\tilde{a} = (p - 1)/2$ and the (0,1) D-brane is transformed into a (1,0) brane. The D-string transforms into the fundamental string with $M \sim 1$, the 3-brane is invariant and now has $M \sim \tilde{g}^{-1}$, the D5-brane transforms to the NS 5-brane with $M \sim \tilde{g}^{-2}$, the D7-brane transforms to the (1,0) 7-brane with $M \sim \tilde{g}^{-3}$, and the D9-brane transforms to the (1,0) 9-brane proposed in [22] with $M \sim \tilde{g}^{-4}$. T-duality can then be used to relate these to branes of the IIA theory.

Consider next $(8, 1^3)$-branes with

$$M = \frac{R_1 \ldots R_8 R_9^3}{l_p^{12}} \,. \tag{3.8}$$

Choosing R_9 to be the string direction corresponding to g_s gives a state with mass

$$M = \frac{R_1 \ldots R_8}{g_s l_s^9} \tag{3.9}$$

and is the D8-brane of type IIA. In [22], it was argued that the D8-brane must come from an M-theory brane which carries a 9-form charge occuring on the right-hand-side of the 11-dimensional superalgebra, so that it is in some ways like a 9-brane. We have thus learned that this M9-brane which gives the M-theory origin of the D8-brane is a $(8, 1^3)$-brane of M-theory. If instead R_1 is chosen as the string direction, one obtains a $(7, 1^3)$-brane with

$$M = \frac{R_2 \ldots R_8 R_9^3}{g_s^3 l_s^{11}} \tag{3.10}$$

while choosing the transverse string direction, X^{10}, gives a $(8, 1^3)$-brane with

$$M = \frac{R_1 R_2 \ldots R_8 R_9^3}{g_s^4 l_s^{12}} \,. \tag{3.11}$$

The $(8, 1^3)$-brane corresponds to the IIA 9-brane proposed in [22].

The $(5, 3^2)$ states with

$$M = \frac{R_1 \ldots R_5 R_6^2 R_7^2 R_8^2}{l_p^{12}} \tag{3.12}$$

first contribute for compactifications of M-theory on an 8-torus. Formally, taking the string theory limit corresponding to shrinking R_1 gives a $(5, 2^2)$-brane with mass proportional to g^{-2}, shrinking R_6 gives a $(4, 3^2)$-brane with mass proportional to g^{-3} and shrinking R_9 gives a $(5, 3^2)$-brane with mass proportional to g^{-4}.

The 8-brane with mass

$$M = \frac{R_1 \ldots R_8}{l_p^9} \tag{3.13}$$

was proposed in [14] and on reducing to string theory gives a 7-brane with tension proportional to $g_s^{-2} l_s^8$ and an 8-brane with tension proportional to $g_s^{-3} l_s^9$ [14]. The $(2, 6^2)$-brane gives a $(2, 6^2)$-brane, a $(1, 6^2)$-brane and a $(2, 5^2)$-brane of type IIA, with masses proportional to $l_s^{-15} g_s^{-5}$, $l_s^{-14} g_s^{-4}$, and $l_s^{-13} g_s^{-3}$, respectively.

The reduction of other branes can be obtained similarly. The T-duality transformation

$$R_i \to \frac{l_s^2}{R_i}, \qquad g_s \to g_s \frac{l_s}{R_i} \tag{3.14}$$

can be used to relate the new type IIA states above to states of the type IIB theory. This doesn't change the dependence on g_s, but does change the dependence on the R_i and the string length. It will be useful to augment the notation above for string theory to include dependence on the string coupling, so that a $(p^\alpha, q^\beta, ..., r^\gamma; x)$-brane has mass proportional to

$$g_s^{-x} (R_{i_1} \cdots R_{i_p})^\alpha (R_{j_1} \cdots R_{j_q})^\beta ... (R_{k_1} \cdots R_{k_p})^\gamma \,, \tag{3.15}$$

with $\alpha < \beta < .... < \gamma$.

The $(5, 1^2; 2)$-brane or IIA KK monopole gives a (5;2)-brane which is the NS 5-brane, a $(5, 1^2; 2)$-brane or type IIB KK monopole, or a $(5, 2^2; 2)$-brane, depending on which dimension is dualised. The $(6, 1^2; 3)$-brane gives a $(7; 3)$-brane, a $(5, 2^2; 3)$-brane and a $(6, 1^2, 1^3; 3)$-brane, depending on which dimension is dualised. As mentioned above, the $(7; 3)$-brane is related by S-duality to the $(7; 1)$-brane or D7-brane, so that these are (1,0) and (0,1) 7-branes. The $(8, 1^3; 4)$-brane or IIA 'NS 9-brane' gives a $(9; 4)$-brane, a $(7, 2^3; 4)$-brane and a $(8, 1^3, 1^4; 4)$-brane, depending on which dimension is dualised. The $(9; 4)$-brane is S-dual to a $(9; 1)$-brane, which is the NS 9-brane proposed in [22], and these are the (1,0) and (0,1) 9-branes of [22].

For M-theory compactified to 3 or less dimensions, there are new states outside the usual brane-scan, most of which do not yet have a string theory or supergravity interpretation. M-theory compactified on T^d has 0-branes in the expected representations of E_d for $d \leq 7$. For $d < 7$, the number of elementary 0-branes (i.e. the dimension of the charge lattice [20]) is equal to the number of vector fields in the supergravity theory and all 0-branes arise from wrapped M2 and M5 branes, together with pp-waves in the internal dimensions. For $d = 7$, $D = 4$ supergravity has 28 vector fields, but 0-branes can carry electric and/or magnetic charge with respect to each, so the charge lattice is 56-dimensional. 49 elementary 0-branes arise from wrapped M2 and M5 branes and pp-waves, while the remaining 7 arise from $(6, 1^2)$-branes or KK monpoles on T^7 (with seven different ways of choosing the Taub-NUT fibre among the 7 compact dimensions). For M-theory on T^8, we have learned that the 0-branes should fit into the **248** representation of E_8, and that there is a matrix model (or rather SYM) construction of all these states. Some of these arise from the expected pp-waves, and wrapped M2,M5 and $(6,1^2)$-branes, but this is not enough to complete the representation. In addition, one needs 8 wrapped 8-branes, 56 wrapped $(5,3^2)$-branes, 28 wrapped $(2,6^2)$-branes and 8 wrapped $(7^2,1^3)$-branes. The counting of 8-branes is mysterious, but was motivated earlier. For $d=9$, the 0-brane spectrum is infinite, with BPS states arising from the wrapped branes discussed above, plus an infinite set of generalisations.

It is interesting to ask whether these BPS states arise as solutions of the supergravity theory. The web of duality relations linking these solutions gives a recipe for their construction, and in general these need not give asymptotically flat solutions (and so will not carry charges occuring in the $D=11$ superalgebra). For example, we have seen that the $(6,1^2;3)$-brane whose mass (3.6) has the unusual g_s^{-3} dependence arises from the M-theory $(6,1^2)$-brane or KK monopole by compactifying one of the transverse directions and identifying this with the string coupling constant. Thus to construct the corresponding supergravity solution, one must take a periodic array of KK monopoles and then identify one of the transverse directions so that the solution is specified by a harmonic function on $\mathbb{R}^2 \times S^1$ instead of the usual multi-Taub-NUT case which is specified by a harmonic function on $\mathbb{R}^3$. Now T-dualising can give a $(7;3)$-brane, a $(5,2^2;3)$-brane or a $(6,1^2,1^3;3)$-brane of the type IIB theory, depending on which direction is dualised. The $(5,2^2;3)$-brane arises from T-dualising one of the directions inside the 6-dimensional world-volume, and the $(7;3)$-brane arises by restricting to solutions which are independent of a transverse direction, and compactifying and T-dualising this direction. The $(p,1^2)$-branes correspond to KK monopole p-brane solutions with the 1^2 corresponding to the Taub-NUT fibre direction. This suggests that (p,q^2)-branes could correspond to spaces with a T^q fibration.

A similar set of states arises for compactification on a d-torus which includes a null circle, and hence for matrix theory, although the wrapped branes can have different properties depending on whether the brane wraps the longitudinal direction or not, and in certain situations the SYM theory is modified to become SYM on a non-commutative torus [25]. For M-theory on $T^d \times S^1$, the 0-branes are arranged into various multiplets of E_{d+1} with different R_{11} dependence; these are the D0-brane (2.10), the flux multiplet in table 2, the momentum multiplet in table 1, the D6-brane multiplet in (2.11), the D8-brane multiplet in (2.12), plus others for $d \geq 8$. On taking the matrix theory large R_{11} limit [1], all multiplets except the flux and 0-brane multiplet have masses that diverge with R_{11} and are truncated out. The flux multiplet is in precisely the correct representation of E_d to give all the 0-branes of M-theory compactified on T^d. For example, the flux multiplet for M-theory on $T^7 = T^6 \times S^1$ (occuring in the $d=7$ row of table 4) is a **56** of E_7, and in the large R_{11} limit gives all 56 0-branes of SYM on $T^6 \times S^1$, which split into a flux multiplet, a momentum multiplet and a D0-brane singlet and D6-brane singlet (occuring in the $d=6$ row of table 4).

In [24], the question was raised as to whether there could be an extra hidden dimension in the matrix theories for $d=7,8$, in the same way as there is for $d=4$. In

$d = 4$, this comes about because there is a 0-brane which combines with the 4 states with momentum in each of the four directions on T^4 to form a **5** of $SL(5)$. On T^d, there is a d-momentum which combines with other 0-branes in the momentum multiplet to form a representation of E_d, and the question arises as to whether this includes a **d+1** of $SL(d+1)$ in the cases in which $SL(d+1)$ is a subgroup of E_d (this is so for $d = 4, 7, 8$); if so, this would suggest that the theory is naturally formulated on T^{d+1} instead of T^d. For $d = 4$, as just stated, the momentum multiplet is a **5** of $E_4 = SL(5)$ and the matrix theory is naturally formulated on $T^5 \times \mathbb{R}$. For $d = 5$, the momentum multiplet is a **10** of $E_5 = SO(5, 5)$, which splits into a $\mathbf{5} + \mathbf{5}'$ of $SL(5)$, corresponding to 5 momentum modes and 5 winding modes of a 6-dimensional string theory. For $d = 6$, the momentum multiplet is a **27** of E_6, which splits into a $\mathbf{6} + \mathbf{21}$ of $SL(6)$, corresponding to 5 momentum modes and 21 membrane wrapping modes on T^6.

For $d = 7$, the momentum multiplet is a **133** of E_7, which splits into a $\mathbf{63} + \mathbf{35} + \mathbf{35}'$ of $SL(8)$, and hence to a $\mathbf{48} + \mathbf{1} + \mathbf{7} + \mathbf{7} + \mathbf{35} + \mathbf{35}'$ of $SL(7)$. The 7 momentum modes do not form part of a **8** of $SL(8)$, so there does not seem to be a natural formulation on T^8. The $\mathbf{35} + \mathbf{35}'$ are naturally associated with 3-brane and 4-brane wrapping modes on T^7, the extra **7** could correspond to string or 6-brane wrapping modes, but the **48** is rather mysterious, suggesting an object coupling to a mixed second-rank tensor $L^i{}_j$ on T^7, instead of the antisymmetric tensor to which a brane couples. A wrapped p-brane with volume form $\omega_{i_1 \dots i_p}$ has mass proportional to

$$\omega^{i_1 \dots i_p} R_{i_1} \dots R_{i_p} \tag{3.16}$$

and the number of distinct states is the number of components of the volume form, while the states that combine with the momentum here have mass

$$\frac{V_R R_{11}}{l_p^9} L^i{}_j \frac{R_i}{R_j} \tag{3.17}$$

and the number of such states is $48 + 1$, the number of components of the tensor $L^i{}_j$. These consist of 42 wrapped KK monopoles plus 7 8-brane states.

For $d = 8$, the momentum multiplet is a **3875** of E_8, which splits into a **80** + **240** + **240'** + **1050** + **1050'** + **1215** of $SL(9)$. The **80** splits into a $\mathbf{63} + \mathbf{1} + \mathbf{8} + \mathbf{8}$ of $SL(8)$, containing the 8-momentum, which does not combine with a 0-brane to give a 9-momentum, but again combines with mysterious tensor states. Progress in the cases $d \geq 7$ will clearly require a better understanding of these 8-branes and other exotic branes.

References

[1] T. Banks, W. Fischler, S. Shenker and L. Susskind, *M theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [`hep-th/9610043`].

[2] For a review see T. Banks, *Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 261 [hep-th/9706168]; *Nucl. Phys.* **67** *(Proc. Suppl.)* (1998) 180 [hep-th/9710231].

[3] L. Susskind, hep-th/9704080.

[4] W. Taylor, *Phys. Lett.* **B 394** (1997) 283 [hep-th/9611042].

[5] L. Motl, hep-th/9701025.

[6] T. Banks and N. Seiberg, *Nucl. Phys.* **B 497** (1997) 41 [hep-th/9702187].

[7] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys.* **B 500** (1997) 43 [hep-th/9703030].

[8] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys.* **B 486** (1997) 77 [hep-th/9603126]; *Nucl. Phys.* **B 486** (1997) 89 [hep-th/9604055]; *Nucl. Phys.* **B 500** (1997) 43 [hep-th/9703030]; *Nucl. Phys.* **B 506** (1997) 121 [hep-th/9704018].

[9] M. Rozali, *Phys. Lett.* **B 400** (1997) 26 [hep-th/9702136].

[10] M. Berkooz, M. Rozali and N. Seiberg, *Phys. Lett.* **B 408** (1997) 105 [hep-th/9704089].

[11] W. Fischler, E. Halyo, A. Rajaraman and L. Susskind, *Nucl. Phys.* **B 501** (1997) 409 [hep-th/9703102].

[12] N. Seiberg, *Phys. Lett.* **B 408** (1997) 98 [hep-th/9705221].

[13] F. Hacquebord and H. Verlinde, *Nucl. Phys.* **B 508** (1997) 609 [hep-th/9707179].

[14] S. Elitzur, A. Giveon, D. Kutasov and E. Rabinbovici, *Nucl. Phys.* **B 509** (1998) 122 [hep-th/9707217].

[15] A. Losev, G. Moore and S.L. Shatashvili, *Nucl. Phys.* **B 522** (1998) 105 [hep-th/9707250];
I. Brunner and A. Karch, *Phys. Lett.* **B 416** (1998) 67 [hep-th/9707259];
A. Hanany and G. Lifschytz, *Nucl. Phys.* **B 519** (1998) 195 [hep-th/9708037];
R. Argurio and L. Houart, *Nucl. Phys.* **B 517** (1998) 205 [hep-th/9710027].

[16] A. Sen, hep-th/9709220.

[17] N. Seiberg, *Phys. Rev. Lett.* **79** (1997) 3577 [hep-th/9710009].

[18] M. Blau and M. O'Loughlin, *Nucl. Phys.* **B 525** (1998) 182 [hep-th/9712047].

[19] C.M. Hull and B. Julia, in preparation.

[20] C.M. Hull and P.K. Townsend, *Nucl. Phys.* **B 438** (1995) 109 [hep-th/9410167].

[21] P.K. Townsend, *Phys. Lett.* **B 350** (1995) 184 [hep-th/9501068].

[22] C.M. Hull, *Nucl. Phys.* **B 509** (1998) 216 [hep-th/9705162].

[23] C.M. Hull, *Nucl. Phys.* **B 468** (1996) 113 [hep-th/9512181].

[24] C.M. Hull, hep-th/9711179.

[25] A. Connes, M.R. Douglas and A. Schwarz, *JHEP* **02** (1998) 003 [hep-th/9711162]; M.R. Douglas and C.M. Hull, *JHEP* **02** (1998) 008 [hep-th/9711165].